Chapter 5

The Wigner-Eckart theorem

We will now prove the Wigner-Eckart theorem. The theorem states that the matrix elements of tensor operators with respect to angular momentum eigenfunctions satisfy

$$\langle \gamma' j' m' | T_q^{(k)} | \gamma j m \rangle = (-1)^{j'-m'} \begin{pmatrix} j' & k & j \\ -m' & q & m \end{pmatrix} \langle \gamma' j' || T^{(k)} || \gamma j \rangle \tag{5.1}$$

$\langle \gamma' j' || T^{(k)} || \gamma j \rangle$ is the reduced matrix element and it is independent of the quantum numbers m, m' and q. The fact that the left side of Eq. (5.1) can be written as a product of the Clebsch-Gordan coefficient and the reduced matrix element has important consequences. The first factor, that does <u>not</u> depend on the nature of the tensor operator, depends only on the geometry of the problem, *i.e.* how the system is oriented with respect to the z-axis. The reduced matrix element on the other hand is independent of the quantum numbers m, m' and q, hence not dependent on the orientation of the system. However, it does of course include for example the dynamics of the system because γ' and γ could include the radial quantum number. When evaluating matrix elements of the type $\langle \gamma' j' m' | T_q^{(k)} | \gamma j m \rangle$ we only need to do so for one set of m, m' and q because the rest can be related *via* the Wigner-Eckart theorem. Clearly this is very helpful.

Following from the properties of the $3j$-symbols (Eq. (2.23)) we immediately see that Eq. (5.1) vanish unless

$$\left. \begin{array}{l} |j-k| \le j' \le j+k \\ m' = m+q \end{array} \right\} \tag{5.2}$$

To prove the Wigner-Eckart theorem we begin with to recall the ladder operator (Eq. (1.8))

$$\left. \begin{array}{l} J_+ |jm\rangle = \sqrt{(j-m)(j+m+1)}\hbar |jm+1\rangle \\ J_- |jm\rangle = \sqrt{(j+m)(j-m+1)}\hbar |jm-1\rangle \end{array} \right\} \tag{5.3}$$

We also need the recursion relation between Clebsch-Gordan coefficients with fixed j_1, j_2 and j but with different m_1 and m_2.

$$J_\pm |j_1 j_2 j m\rangle = (J_{1\pm} + J_{2\pm}) \sum_{m_1 m_2} |j_1 j_2 m_1 m_2\rangle \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \tag{5.4}$$

Eqs. (5.3) now give (set $m_1 = m_1'$ and $m_2 = m_2'$)

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m + 1)}|j_1 j_2 j m \pm 1\rangle = \\
&\sum_{m_1' m_2'} (\sqrt{(j_1 \mp m_1')(j_1 \pm m_1' + 1)}|j_1 j_2 m_1' \pm 1 m_2'\rangle + \\
&\sqrt{(j_2 \mp m_2')(j_2 \pm m_2' + 1)}|j_1 j_2 m_1' m_2' \pm 1\rangle)\langle j_1 j_2 m_1' m_2'|j_1 j_2 j m\rangle
\end{aligned} \tag{5.5}$$

Multiplying with $\langle j_1 j_2 m_1 m_2|$ from the left in Eq. (5.5) and noting that the first term on the right side is non-zero only if $m_1 = m_1' + 1$, $m_2 = m_2'$ and that the second term on the right side of Eq. (5.5) is non-zero only if if $m_1 = m_1'$ and $m_2 = m_2' + 1$. We now arrive at the recursion relation for the Clebsch-Gordan coefficients,

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m + 1)}\langle j_1 j_2 m_1 m_2|j_1 j_2 j m \pm 1\rangle = \\
&\sqrt{(j_1 \mp m_1)(j_1 \pm m_1)}\langle j_1 j_2 m_1 \mp 1 m_2|j_1 j_2 j m\rangle + \\
&\sqrt{(j_2 \mp m_2)(j_2 \pm m_2)}\langle j_1 j_2 m_1 m_2 \mp 1|j_1 j_2 j m\rangle +
\end{aligned} \tag{5.6}$$

Now consider (using Eq. (4.20))

$$\langle \gamma' j' m'|[J_\pm, T_q^{(k)}]|\gamma j m\rangle = \hbar\sqrt{(k \mp q)(k \pm q + 1)}\langle \gamma' j' m'|T_{q\pm 1}^{(k)}|\gamma j m\rangle \tag{5.7}$$

or equivalently, expanding the commutator and using Eq. (5.3), we get

$$\begin{aligned}
&\sqrt{(j' \pm m')(j' \mp m' + 1)}\langle \gamma' j' m'_{\mp 1}|T_q^{(k)}|\gamma j m\rangle = \\
&\sqrt{(j \mp m)(j \pm m + 1)}\langle \gamma' j' m'|T_q^{(k)}|\gamma j m \pm 1\rangle + \\
&\sqrt{(k \mp q)(k \pm q + 1)}\langle \gamma' j' m'|T_{q\pm 1}^{(k)}|\gamma j m\rangle
\end{aligned} \tag{5.8}$$

substitute $j' \to j$, $m' \to m$, $j \to j_1$, $m \to m_1$, $k \to j_2$ and $q \to m_2$ we get with $\pm \to \mp$

$$\begin{aligned}
&\sqrt{(j \mp m)(j \pm m + 1)}\langle \gamma' j m \pm 1|T_{m_2}^{(j_2)}|\gamma j_1 m_1\rangle = \\
&\sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)}\langle \gamma' j m|T_{m_2}^{(j_2)}|\gamma j_1 m_1 \mp 1\rangle + \\
&\sqrt{(j_2 \mp m_2)(j_2 \pm m_2)}\langle \gamma' j m|T_{m_2 \mp 1}^{(j_2)}|\gamma j_1 m_1\rangle
\end{aligned} \tag{5.9}$$

A quick glance at Eqs. (5.6) and (5.9) soon relieves that the two equations are very similar. In fact, by rewriting the two equations as

$$\left.\begin{aligned}
ax + by + cz = 0 \\
ax' + by' + cz' = 0
\end{aligned}\right\} \tag{5.10}$$

where

$$\left.\begin{aligned}
a &= -\sqrt{(j \mp m)(j \pm m + 1)} \\
b &= \sqrt{(j_1 \mp m_1 + 1)(j_1 \pm m_1)} \\
c &= \sqrt{(j_2 \mp m_2 + 1)(j_2 \pm m_2)}
\end{aligned}\right\} \tag{5.11}$$

one sees that Eq. (5.10) describe two parallel planes, *i.e.*

$$\frac{x}{x'} = \frac{y}{y'} = \frac{z}{z'} = \text{constant} \tag{5.12}$$

Eq. (5.12) is obviously true for all components and by picking one of them, *e.g.* the second we get

$$\frac{\langle \gamma' j m | T^{(j_2)}_{m_2} | \gamma j_1 m_1 \mp 1 \rangle}{\langle j_1 j_2 m_1 \mp 1 m_2 | j_1 j_2 j m \rangle} = \text{constant} \tag{5.13}$$

and with the substitutions $j \to j'_1$, $m \to m'$, $j_2 \to k$, $m_2 \to q_1$, $j_1 \to j$ and $m_1 \mp 1 \to m$ we have

$$\langle \gamma' j' m' | T^{(k)}_q | \gamma j m \rangle = \text{constant} \cdot \langle j k m q | j k j' m' \rangle \tag{5.14}$$

and the theorem has been proven! Once again we note that the constant is independent of m, m' and q and that all geometrical dependencies are in the Clebsch-Gordan coefficient ($\propto$ $3j$-symbol). Our problem has been separated into two parts, one with and the other without geometrical dependence.